A THEOREM CONCERNING EQUAL RATIOS

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The following simple theorem is not included in the common textbooks on the Theory of Numbers, but it is altogether likely that it has been published before. The writer would be grateful to any reader of The Annals who should tell him where a demonstration has already appeared.

Theorem: If a set of ratios between positive integers are equal to one another, then all are equal to the ratio of the greatest common divisor of the numerators to that of the denominators, and also to the ratio of the least common multiple of the numerators to that of the denominators.

To prove the theorem, let

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \cdots = \frac{a_n}{b_n} = \frac{p}{q}$$

where p and q are any two positive integers in the same ratio as the positive integers, a_i and b_i where $i = 1, 2, \cdots n$. We have then n equations

$$qa_i = pb_i.$$

The greatest common divisor of the first members of these equations must be equal to that of the second members, so that if a be the greatest common divisor of the a_i's and b that of the b_i's, then, since qa and pb are the greatest common divisors of the first and second members respectively, we have

$$qa = pb \qquad \text{or} \qquad \frac{a}{b} = \frac{p}{q}.$$

If, further, we put

$$a_i = r_i a \qquad b_i = s_i b$$

then, since

$$\frac{a_i}{b_i} = \frac{a}{b}$$

we have $r_i = s_i$.

Now if A and B are the least common multiples of the a_i's and b_i's respectively, and if R and S are the least common multiples of the r_i's and s_i's respectively, we have

$$A = Ra \qquad B = Sb$$

and since, for all values of i, $r_i = s_i$, $R = S$, and consequently

$$\frac{A}{B} = \frac{a}{b} = \frac{p}{q}$$

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